MTH 305: Practice assignment 1

1 Countability

(i) Let S be an infinite set. Show that the following statements are equivalent.

(a) S is countably infinite.

(b) There exists a surjective function $f : \mathbb{N} \to S$.

(c) There exists a bijective function $g : S \to \mathbb{N}$.

(ii) Show that a countable union of countable sets is countable.

(iii) Show that a finite cartesian product of countable sets is countable. Is a countably infinite cartesian product of countable sets countable? Explain why or why not.

(iv) Show that if S is an uncountable set and $T \subset S$ is a countable set, then $S \setminus T$ is uncountable. Use this to show that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

(v) The set of infinite sequences of binary digits (i.e. 0 and 1) is uncountable.

(vi) Show that the power set of a countably infinite set is uncountable.

2 Mathematical induction

Establish the following using the first (or second) principle of mathematical induction.

(i) $1 + 3 + \ldots + (2n - 1) = n^2$, for $n \geq 1$.

(ii) $1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$, for $n \geq 1$.

(iii) $1^3 + 2^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$, for $n \geq 1$.

(iv) $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \ldots + a + 1)$.

(v) $n! > n^3$, for $n \geq 6$.

(vi) $(1+a)^n \geq 1 + na$, for $n \geq 1$, whenever $a > -1$.

(vii) If $a_1 = 11$, $a_2 = 21$, and $a_n = 3a_{n-1} - 2a_{n-2}$, for $n \geq 3$, then for $n \geq 1$,

$$a_n = 5 \cdot 2^n + 1.$$

(viii) $\dfrac{1}{2} + \dfrac{2}{2^2} + \ldots + \dfrac{n}{2^n} = 2 - \dfrac{n+2}{2^n}$.

3 Binomial theorem

Establish the following assertions.

(i) The Binomial theorem.

(ii) $\dbinom{n}{k} = \dfrac{n-k+1}{k}\dbinom{n}{k-1}$, for $n \geq k \geq 1$.

(iii) $\dbinom{n}{k} = \dbinom{n-2}{k-2} + 2\dbinom{n-2}{k-1} + \dbinom{n-2}{k}$, for $2 \leq k \leq n-2$ and $n \geq 4$.

(iv) $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

(v) $\displaystyle\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

(vi) $\displaystyle\sum_{k=1}^{n}\binom{2k}{2} = \frac{n(n+1)(4n-1)}{6}$, for $n \geq 2$.

(vii) $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} = \frac{1}{n+1}$.

(viii) $\displaystyle\binom{n}{r} = \binom{n}{r+1}$ if and only if n is odd and $r = \frac{n-1}{2}$.

(ix) $2^n < \binom{2n}{n} < 2^{2n}$, for $n > 1$.